Buyer-optimal Matching in Two-Sided Platforms*

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Abstract

A platform matches a unit-mass of sellers, each owning a single product of heterogeneous quality, to a unit-mass of buyers with differing valuations for unit-quality. After the matching, sellers make take-it-or-leave-it price-offers to buyers. Initially the valuations of buyers are only known to them and the platform, but sellers make inference from the matching algorithm. We show that, while the efficient matching is positive-assortative, buyer-optimal matchings are, often, stochastically negative-assortative (i.e., compared to lower-quality sellers, high-quality sellers are matched to buyers with lower expected valuation). We highlight how distorting a matching to favor the side lacking bargaining power may result in inefficiencies.

KEYWORDS: Two-sided markets, matching, monopoly, information leakage, platforms.

JEL CLASSIFICATION: D82, D42, D47, C78

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1 Introduction

Digital platforms have been very successful at intermediating a wide range of human activities, mainly due to their ability at minimising a variety of transaction costs at scale. In thick and congested markets, *identifying most relevant opportunities* is one such cost, which platforms reduce by algorithmically matching their users to selected subsets of options. For instance, Google Shopping uses the enormous amount of data it collects about users in order to expose consumers to a prominent subset of businesses, often chosen among thousands of potential ones. An analogous service is offered by Amazon, eBay and Facebook Marketplace which recommend products to users who have filled a search query. Similarly, for popular destinations, Airbnb does not provide on-screen to lodgers all available rental options, but a tailored sample.

Of course, while the choice of the matching algorithm depends on the platform's incentives, outcomes are also driven by user behavior. For example, Google may display more prominently high-quality purchase opportunities to consumers that it may know to have higher willingness-to-pay.[1] As a result, it is conceivable that sellers base their prices not just on their quality and prior-information, but also on the sample of buyers that they usually interact with. Then, in light of the non-trivial interplay of matching algorithm and user behavior, the questions arise as to what extent platforms make choices that induce socially optimal outcomes *and* how to engineer such algorithms for the benefit of consumers, should regulation become warranted.[2]

In this paper we contribute toward answering these questions by looking at properties of the buyer-optimal matching algorithm in an environment where sellers with quality-differentiated products price monopolistically after being matched with privately-informed buyers.The matching not only determines the available gains from trade, but also reveals information to firms in a way that ultimately affects their market power vis-a-vis consumers. Our main observation is that consumers, as an aggregate, often benefit form a heavily distorted, often negative-assortative, matching because, by concealing information about consumers' valuations, such a matching limits firms' market power and keeps prices low.

While the focus on buyer-optimal matching has, primarily, a normative motivation, there are several plausible scenarios where maximization of buyer-surplus is a good proxy for the incentives of a two-sided platform. For example, two platforms might be in direct competition to attract buyers, because sellers face switching costs or have already sunk investments that lock them in one or the other. Also, a monopolist platform may be constrained regarding its ability to charge one side, which may lead the platform to maximize the surplus of the side that can be charged; or a platform might earn from advertising to one side only and therefore attempt to maximize participation on that side.

One policy implication of our analysis is that, rather than by tampering with the matching algorithm, buyer-surplus might be more effectively raised by altering bargaining power between the two sides of the market.[3] Letting buyers make proposals to sellers is, indeed, an existing business model in platform

[1] Businesses bid based on information provided by Google. High-quality business are likely to bid more for high-valuation consumers. Another case is that of Orbitz, an online travel agent that used to show to Mac users more expensive hotels than those it showed to Windows users. See Dana Mattioli, "On Orbitz, Mac Users Steered to Pricier Hotels", Wall Street Journal (2012).

[2] The view that big tech platforms, such as Google and Amazon, should be regulated as public utilities is popular. The current chair of the US FTC has expressed this view in her landmark paper on Amazon, see Khan (2016). In Europe a draft regulation, the Digital Market Act, is being discussed and if approved it would impose specific behavior on systemically important platforms, deemed "gatekeepers". Among such rules is one that forbids platforms from distorting their algorithms in favor of own products, a practice called "self-preferencing".

[3] Of course, a platform who knows and controls everything, would match users efficiently and enforce the desired prices. This

markets. Such policy has been popularized by Priceline, an online travel agent, and since then has been widely adopted, quite possibly in an attempt to increase buyers' surplus (e.g., EBay now allows buyers to make offers to consenting sellers).

We now proceed with presenting the model and our results. Formal proofs are in Appendix. We discuss the related literature in the concluding section of the paper.

2 The Model

There is a unit mass of buyers. Each buyer has either low valuation, $l\,(>0)$, or high valuation, $h\,(>l)$. The fraction of buyers with high valuation is μ. There is also a unit mass of sellers. Each seller has a single good to sell. The quality of a seller is q and distributed according to the atomless CDF F with support in $[\underline{q},\overline{q}]$. If a buyer with valuation $v\,(\in\{l,h\})$ purchases the good from a seller with quality q at price t, the buyer's payoff is $vq-t$ and the seller's payoff is t. The buyers and sellers are matched by a platform. We assume that the platform and the buyer know the buyer's valuation but the seller does not.[4] Without loss, we assume that sellers'qualities are common knowledge.[5] Once a buyer and a seller are matched, the seller makes a take-it-or-leave-it price-offer to the buyer.[6] If the buyer accepts the seller's offer, they trade at the price set by the seller. Otherwise each of them gets their reservation payoff of zero.

Matching.— We describe a *matching* by the probabilities that each seller q is matched with a buyer with valuations h or l. That is, a matching is given by measurable $p=(p_h,p_l)$, $p_h,p_l:\left[\underline{q},\overline{q}\right]\to[0,1]$, where $p_h\,(q)$ and $p_l\,(q)$ denotes the probabilities that a q-seller is matched with a buyer with valuations h and l, respectively. A *feasible* matching p must satisfy the following constraints:

$$\begin{aligned} p_h\,(q)+p_l\,(q) &\leq 1, \\ \int_{\underline{q}}^{\overline{q}} p_h\,(q)\,dF\,(q) &\leq \mu, \\ \int_{\underline{q}}^{\overline{q}} p_l\,(q)\,dF\,(q) &\leq 1-\mu. \end{aligned}$$

The first constraint guarantees that the probability that a seller with type q is matched with a buyer is weakly less than one. We do not require that each buyer and seller is matched with probability one. The second and third constraints guarantee the measure of high-value (low-value) buyers who are matched with sellers does not exceed the total measure of high-value (low-value) buyers.

We say that a matching p is *positive-assortative*, if $p_h(q)=1$ for $q\geq F^{-1}(1-\mu)$ and $p_h(q)=0$ elsewhere, where F^{-1} stands for the inverse of F. We say that the matching is *stochastically negative-assortative* whenever p_h is monotonically non-increasing in quality. Conversely, the matching is *stochastically positive-assortative* when p_h is monotonically non-decreasing. A matching is *fully-random* if and

scenario might be a closer representation of the business model of Uber but it is not the norm.

[4]The assumption that platforms have more information about their users on one side of the market than that held by users on the other side of the same market seems, in many cases such as that of Google and Facebook, plausible.

[5]The assumption that the quality of sellers become known is without loss because a platform that maximizes buyer-surplus will always make the value of the seller known to buyers.

[6]While factual in many cases, the assumption that firms cannot commit to a price before being matched also remains valid in settings where the price is posted in advance, as long as the matching algorithm does not heavily rely on the price-variable, as it seems to be the case for many platforms. For instance Ebay's Best Match (i.e., the search algorithm) considers, in addition to the price, whether the listing matches the buyer's search term, how popular the item is, the quality of the listing, how complete it is, the listing terms of services and the track record of the seller.

only if $p_h(q) = \mu$ and $p_l(q) = 1 - \mu$ for all $q \in [\underline{q}, \overline{q}]$.

Optimal Prices.— If the matching is given by $p = (p_h, p_l)$, the posterior probability of seller with quality q that she is matched with a high-value buyer is $\mu^p(q) = p_h(q) / (p_h(q) + p_l(q))$. So, the q-quality seller is willing to set price qh if, and only, if $\mu^p(q) \geq l/h$. Such seller is willing to set price ql if, and only if, $\mu^p(q) \leq l/h$. Because we are interested in maximizing buyer-surplus, it is without loss of generality and it simplifies notation to assume that, when a seller is indifferent between prices, it charges the lowest.

Buyer Surplus.— Let $\chi^p(q) \in [0,1]$ denote the probability that a seller with value q charges price lq following matching p. That is

$$\chi^p(q) = \begin{cases} 1 & \text{if } \mu^p(q) \leq l/h, \\ 0 & \text{if } \mu^p(q) > l/h. \end{cases}$$

Then, for given χ^p, the buyers' surplus can be expressed as

$$\int_{\underline{q}}^{\overline{q}} \chi^p(q)\, p_h(q)\, q\,(h-l)\, dF(q).$$

Before proceeding to the results, the assumption that matching is one-to-one deserves motivation. Primarily, it implies that buyers cannot search across multiple sellers. Thus, we abstract from price-competition, focusing on the matching technology and its implications for post-match bargaining.[7] Yet, we believe the assumption is often a good approximation. First, even when consumers are exposed to multiple firms, the top-ranked enjoys a substantial advantage as many consumers may be reluctant to search. It is known that consumers are heavily biased toward most prominently located opportunities, such as the Buy-box placement in Amazon or a top place in Google's ranking (see Narayanan and Kalyanam (2015)), placements for which firms are willing to pay high prices. Second, suppose the algorithm created a ranking of all firms and consumers could continue their search beyond their first match at a cost. Search might still be limited in equilibrium and the first firm could enjoy market power, as illustrated by Diamond (1971).

3 Buyer-Optimal Matching

Our goal is to characterize the matching which maximizes buyers'surplus. We start with observing that this goal is incompatible with that of welfare maximization, where *welfare* is intended as the sum of expected buyers'surplus and sellers'profit. Following Becker (1975), the following result should not be surprising.

Proposition 1. *A matching maximizes total welfare if and only if it is a positive-assortative matching (PAM) almost-everywhere. In the PAM buyers obtain zero surplus.*

A formal proof relies on two observations. First, for any matching p that induces *complete-information*, that is such that $\mu^p(q) \in \{0,1\}$, trade will take place with probability one and sellers will obtain all surplus. Second, PAM induces complete information and the total welfare of a match between a seller with quality q and a buyer with value v is given by the supermodular function qv.

[7]The assumption also implies that each seller has only one product for sale. While product heterogeneity and scarcity are often crucial elements of the markets we are modeling, at the purpose of the analysis we can always treat a seller with multiple units of a product as multiple identical sellers.

Having noted that a buyer-surplus maximizing matching must generate inefficiencies, we now fully characterize the buyer-optimal matching and pricing and show that such inefficiencies can be sizable. We focus on *Pareto-efficient* buyer-optimal matching, that is, there is no other matching that gives higher sellers' surplus without reducing buyer-surplus. This has one main implication: no buyer or seller remains unmatched, even if additional matches do not increase buyer surplus.

Theorem 1. *Let $p^* = (p_h^*, p_l^*)$ be a matching defined as follows:*

$$p_h^*(q) = \begin{cases} l/h & \text{if } q \geq q^* \\ 0 & \text{if } q \leq q^* \text{and } \mu \leq l/h \\ 1 & \text{if } q \leq q^* \text{and } \mu \geq l/h \end{cases},$$

$$p_l^*(q) = 1 - p_h^*(q),$$

$$\text{where } q^* = \begin{cases} F^{-1}\left(\frac{l-\mu h}{l}\right) & \text{if } \mu \leq l/h \\ F^{-1}\left(\frac{\mu h - l}{h-l}\right) & \text{if } \mu \geq l/h \end{cases}.$$

Any Pareto-efficient matching that maximizes buyer-surplus is equal to p^* almost everywhere.

The efficient buyer-optimal matching (henceforth also BOM) of high-value buyers, p_h^*, is exemplified in the two panels below. Observe that, when $\mu \geq l/h$ the matching is stochastically negative assortative. Instead, when $\mu < l/h$ then the matching is stochastically positive assortative.

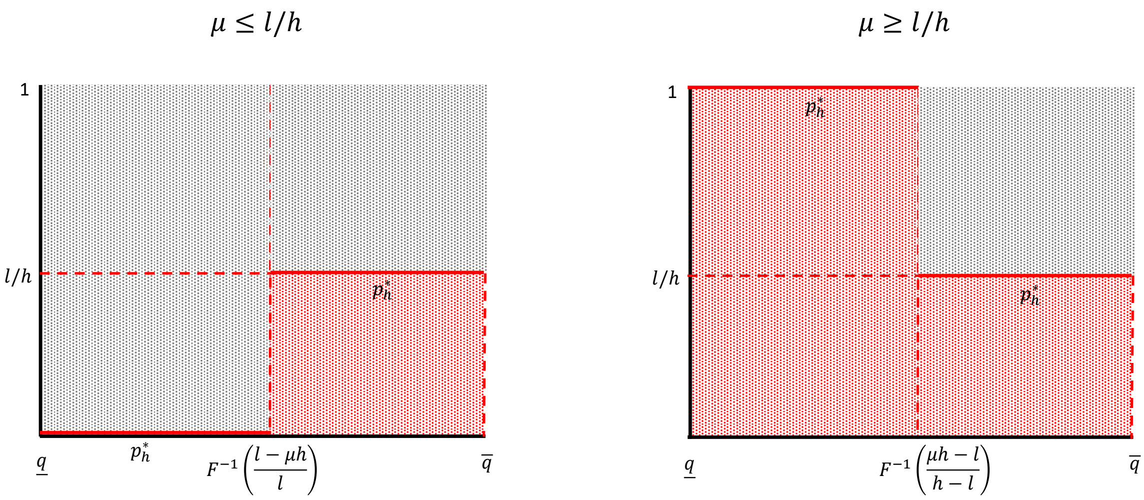

Figure 1: Sketch of Efficient Buyer-Optimal matching, p_h^*.
The red-dotted area has measure μ and the gray-dotted area has measure $1-\mu$.

Let us explain the arguments leading to the statement of this proposition. Recall that a q-seller sets price qh if the probability of being matched with a high-value buyer exceeds l/h and sets price ql otherwise. So, a buyer's payoff is posititive only if his valuation is high and the seller sets the lower of these prices. The surpluss of a high-value buyer who purchases a q-quality good at price ql is $q(h-l)$. Therefore, in order for a seller to generate positive consumer surplus, she must be matched with a mixture of high- and low-value buyers. Furthermore, consumer surplus is maximized at a given seller if she is matched with as many high-value buyers as possible as long as she is willing to set the lower price. That is, she is indifferent between the two prices, so the fraction of high-value buyers among her matches is exactly l/h. Since the high-value buyers' surplus, $q(h-l)$, is increasing in q, the BOM generates consumer surplus at high quality sellers. Above a quality threshold, q^*, q-sellers are matched so that they are indifferent between the two prices, so $p_h^*(q) = 1 - p_l^*(q) = l/h$. Of course, to maximize consumer surplus, the threshold q^* should be as low as possible and it is determined by the initial distribution of the buyers, μ. If low-value buyers are abudants, $\mu \leq l/h$, then q^* is defined so that the fraction of high-value buyers who are matched with sellers with quality above q^* is exactly μ. In this case, sellers with quality below q^* are matched with the remaining low-value buyers. If there are few low-value buyers, $\mu \geq l/h$, then q^* is defined so that the fraction of low-value buyers who are matched with sellers with quality above q^* is exactly $1-\mu$. In this case, sellers belowe q^* are matched with the remaining high-quality buyers.

Welfare analysis. We now compare BOM's payoffs with those arising under two benchmarks, the PAM and the fully random matching (henceforth also FRM). PAM corresponds to the matching that maximizes welfare and profit, as Proposition 1 states. FRM corresponds to the focal case in which the platform does not condition the matching on any information about buyers.

It is obvious that BOM generates strictly larger buyer surplus than PAM and FRM. Regarding the remaining welfare variables, we have the following proposition. The main insight from this welfare comparison is that, in some cases, BOM generates lower welfare than if the platform ignored information altogether.

Proposition 2. *(i) PAM generates higher welfare and higher profit than BOM; (ii) If $\mu \leq l/h$, then FRM generates the same profit as BOM and, hence, lower welfare. If $\mu > l/h$ then FRM generates higher-profit than BOM but the welfare ranking between the two matchings is, in general, ambiguous; when μ is sufficiently large, that is $\mu \geq \frac{l}{h} + \frac{h-l}{h}\frac{l}{h}$, then the welfare of BOM is larger than that of FMR.*

Part (i) is an immediate corollary of Proposition 1. It is also immediate to see that when $\mu \leq l/h$ profit of BOM and FRM are the same. In both cases, each seller sells at price l and trade takes place. Hence, because BOM generates higher buyer-surplus by Theorem 1, then the first part of (ii) follows. The proof in the Appendix shows that, when $\mu > l/h$, profits are lower under BOM than under FRM and shows, by means of two non knife-edge examples, that the welfare comparison between BOM and FRM is ambiguous.

4 Related Literature

A number of other papers have looked at the incentives of platforms who charge per-click to distort matching in order to boost costly search (i.e., clicking). In Eliaz and Spiegler (2011) a consumer searches from a pool of firms whose boundary is determined by the platform. Inefficiently too many low-quality sellers may be allowed in the pool, to induce consumers to search more. In De Corniere (2016) a platform can match consumers to their preferred segment of firms in a more or less noisy way. Consumers search within the set of firms they are matched with. Because perfect matching traps consumers into monopoly prices, a platform may want to bias the algorithm to foster consumer participation. In a similar vein, in Hagiu and Jullien (2011) an intermediary may match consumers with stores that are worse for them in order to persuade them to search more. They also show that the intermediary gains from marginally biasing the matching away from the perfect one, if it reduce firms' prices and convinces more consumers to visit at least one store. While focusing on platform's incentives, these papers suggest reasons why perfect matching may not be consumer-optimal, with De Corniere (2016) and Hagiu and Jullien (2011) also identifying a feedback effect of matching on prices. In contrast, we abstract from horizontal differentiation and competition. The additional simplicity allows us to fully characterise the buyer-optimal matching technology, thus highlighting a trade-off between matching efficiency and consumers'information rent.[8]

In searching for a buyer-optimal matching, a designer resolves a trade-off between a more efficient matching and information rent to buyers. This trade off recurs in other contexts that also share with us a flexible information-design approach. In Armstrong and Zhou (2021), perfectly informing consumers about which of two differentiated products is best for them relaxes competition but maximizes welfare, while the consumer-optimal information structure dampens differentiation to some extent. In a search model, Dogan and Hu (2021) show that total welfare would be maximized by giving a buyer as much information as possible to find a good match among several firms, but that would lead to too little competition within firms. While the focus of the latter two papers is competition between firms, the focus of this paper is on the allocation of access to buyers for monopolistic firms.[9]

There is a vast literature that studies two-sided matching, with pioneering contributions from Gale and Shapley (1962), Shapley and Shubik (1971) and Becker (1973). Most of this literature focuses on characterizing equilibrium matchings, typically imposing a more stringent stability condition. When utility is transferable, or partially so, stability jointly constraints both the matching and how the match-surplus is shared. In contrast, we are *not* concerned with stable matches, as the matching in our model is enforced by a platform. We also depart from tradition by considering an environment where, due to the informational spillovers, the surplus-sharing possibility frontier of a matched couple is not exogenous, but depends on the entire matching.

Intuitively, the more heterogeneous are sellers, the more it is costly to move away from PAM. In fact, if all sellers are the same, then our problem is isomorphic to that of identifying the segmentation of buyer's demand that maximizes buyer-surplus under a price-discriminating monopolist. In this case, it follows from Bergemann et al. (2015) that there exists a matching that it is both efficient and where buyers obtain all surplus except for the monopoly profit from a random matching. While this happens to be true in the binary-valuation case, the matching problem with heterogeneous quality cannot be reduced to that of taking their *uniform profit preserving extremal segmentation*, ordering the segments in terms of average consumer surplus, and then matching sellers to group of buyers in a positive assortative fashion. In fact, as it turns out, the optimal matching with more than two values (i.e., the buyer-optimal segmentation of buyers) may deliver higher than monopoly profit to a monopolist seller and it is thus not a *uniform profit preserving*.[10]

[8]A flexible matching technology has also been studied by Gomes and Pavan (2016) and Jullien and Pavan (2019). In their work, many-to-many matching of two sides of the market is centralized, subject to a tariff, and aimed at being incentive compatible and maximizing welfare in an environment where values are private information.

[9]In Condorelli and Szentes (2020) a buyer can invest in shaping her distribution of value for the product of a seller with bargaining power. Therefore, it faces a related trade-off between having a higher valuation and larger information rent. In both this and our previous paper, information is a byproduct of choices (i.e., distribution of value and matching) that affect both the information structure and feasible surplus. In Armstrong and Zhou (2021) and in Dogan and Hu (2021), the information structure is a primitive that affects agents'choices which may be inefficient.

[10]This can be illustrated by considering a model where consumers have three valuations. Suppose the three possible valuations of buyers are $(1/8, 1/2, 1)$, with the medium and high value having both probability 1/5, and the quality of sellers is either high, with probability 2/5, or low. The buyer-optimal matching in this example requires the creation of an extremal segment, in the

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sense of Bergemann et al. (2015), that contains both medium and high valuation buyers, to be matched with all high-quality seller. Then, the remaining segment of buyers is only composed by low valuation buyers and it is matched to low-quality sellers. It can then be observed that this segmentation, in the case of constant unit-quality studied by Bergemann et al. (2015), delivers a profit of 11/40, which is higher than than the monopoly profit of 1/5.

Appendix: Proofs

Proof of Proposition 1. Denote with G the distribution of buyers'values. Since vq is supermodular, a classic result from G.G. Lorentz (1953) implies that $\sup_{\pi \in \mathscr{M}(F,G)} \mathbb{E}_\pi[vq]$, where $\mathscr{M}(F,G)$ is a *coupling* of probabilities F and G, has a unique *comonotone* solution which is given by PAM. We have already argued the second statement follows from the fact that PAM induces complete information. $\square$

Proof of Theorem 1. Focusing on the case $\mu \geq l/h$, we show that p^* generates strictly larger consumer surplus than p unless $p = p^*$ almost everywhere. The case $\mu \leq l/h$ is analogous and we omit the proof.

For each p, let us defined the CDF G^p as follows:

$$G^p(x) = \frac{\int_{\underline{q}}^{x} \chi^p(q)\, p_h(q)\, dF(q)}{\int_{\underline{q}}^{\overline{q}} \chi^p(q)\, p_h(q)\, dF(q)}.$$

Also, define q^p by

$$\int_{q^p}^{\overline{q}} \frac{l}{h} dF(q) = \int_{\underline{q}}^{\overline{q}} \chi^p(q)\, p_h(q)\, dF(q). \tag{1}$$

Finally, define the CDF H^p by $H^p(x) = 0$ if $x \leq q^p$ and by

$$H^p(x) = \frac{\int_{q^p}^{x} \frac{l}{h} dF(q)}{\int_{q^p}^{\overline{q}} \frac{l}{h} dF(q)}$$

if $x > q^p$.

We now show that H^p first-order stochastically dominates G^p. To see this, first note that if $x \leq q^p$ then $H^p(x) = 0 \leq G^p(x)$. Moreover, for all $x > q^p$,

$$1 - H^p(x) = \frac{\int_{x}^{\overline{q}} \frac{l}{h} dF(q)}{\int_{q^p}^{\overline{q}} \frac{l}{h} dF(q)} = \frac{\int_{x}^{\overline{q}} \frac{l}{h} dF(q)}{\int_{\underline{q}}^{\overline{q}} \chi^p(q)\, p_h(q)\, dF(q)} \geq \frac{\int_{x}^{\overline{q}} \chi^p(q)\, p_h(q)\, dF(q)}{\int_{\underline{q}}^{\overline{q}} \chi^p(q)\, p_h(q)\, dF(q)} = 1 - G^p(x),$$

where the first and last equalities are the definitions of H^p and G^p, respectively, the second equality follows from (1) and the inequality follows from $\chi^p(q) \leq 1$ and $p_h(q) \leq l/h$ whenever $\chi^p(q) = 1$. Then previous inequality chain implies that $H^p(x) \leq G^p(x)$ even when $x > q^p$.

Therefore,

$$\begin{aligned}
& \int_{\underline{q}}^{\overline{q}} \chi^p(q)\, p_h(q)\, q\,(h-l)\, dF(q) = \left[\int_{\underline{q}}^{\overline{q}} \chi^p(q)\, p_h(q)\, dF(q)\right] \left[\int_{\underline{q}}^{\overline{q}} q\,(h-l)\, dG^p(q)\right] \\
= \; & \left[\int_{q^p}^{\overline{q}} \frac{l}{h} dF(q)\right] \left[\int_{\underline{q}}^{\overline{q}} q\,(h-l)\, dG^p(q)\right] \leq \left[\int_{q^p}^{\overline{q}} \frac{l}{h} dF(q)\right] \left[\int_{\underline{q}}^{\overline{q}} q\,(h-l)\, dH^p(q)\right] \\
= \; & \int_{q^p}^{\overline{q}} \frac{l}{h} q\,(h-l)\, dF(q),
\end{aligned}$$

where the first and last equalities follows from the definitions of G^p and H^p, respectively. The second equality is implied by (1) and the inequality follows from the fact that H^p first-order stochastically dominates G^p.

It remains to show that

$$\int_{q^p}^{\overline{q}} \frac{l}{h} q(h-l)\, dF(q) \leq \int_{q^*}^{\overline{q}} \frac{l}{h} q(h-l)\, dF(q).$$

In order to do so, it is enough to argue that $q^p \leq q^*$. Observe however that

$$\begin{aligned} \int_{q^*}^{\overline{q}} \left(1-\frac{l}{h}\right) dF(q) = 1-\mu \geq \int_{\underline{q}}^{\overline{q}} \chi^p(q)\, p_l(q)\, dF(q) \geq \frac{h-l}{l} \int_{\underline{q}}^{\overline{q}} \chi^p(q)\, p_h(q)\, dF(q) \\ = \frac{h-l}{l} \int_{q^p}^{\overline{q}} \frac{l}{h} dF(q) = \int_{q^p}^{\overline{q}} \left(1-\frac{l}{h}\right) dF(q), \end{aligned}$$

where the first equality is the definition of q^*, the first inequality is a feasibility constraint for the matching p. The second inequality follows from the fact that if $\chi^p(q)$ then $p_h(q)\, h \leq (p_l(q) + p_h(q))\, l$, that is, $p_l(q) \geq p_h(q) [(h-l)/l]$. The second equality is again implied by (1). □

Proof of Proposition 2. Profit under the FRM matching is $\mu h \mathbb{E}[q]$. If $\mu > l/h$, then the profit under BOM is smaller than that under FRM if

$$F(q^*)\mathbb{E}[q \mid q \leq q^*]\, h + (1-F(q^*)) \frac{l}{h} \mathbb{E}[q \mid q > q^*]\, h < \mu h \mathbb{E}[q]$$

where we have used the fact that albeit sellers with $q \geq q^*$ charge l, they are indifferent between charging l and h. Dividing both sides by μ, simplifying h away and rewriting $\mathbb{E}[q]$ we get

$$\frac{F(q^*)}{\mu} \mathbb{E}[q \mid q \leq q^*] + \frac{1-F(q^*)}{\mu} \frac{l}{h} \mathbb{E}[q \mid q > q^*] < \mathbb{E}[q] = F(q^*)\mathbb{E}[q \mid q \leq q^*] + (1-F(q^*))\mathbb{E}[q \mid q > q^*].$$

Now focus on the inequality between the left and right side of the above. Since $q^* = F^{-1}\left(\frac{\mu h - l}{h-l}\right)$, we have

$$\frac{F(q^*)}{\mu} + \frac{1-F(q^*)}{\mu} \frac{l}{h} = 1.$$

Hence, both sides are weighted sums of the same conditional expectations. Then, to conclude the proof of the statement that profit under BOM is below profit with FRM observe that

$$\mathbb{E}[q \mid q > q^*] > \mathbb{E}[q \mid q \leq q^*]$$

and, because $\mu \geq l/h$,

$$\frac{F(q^*)}{\mu} > F(q^*) \text{ and } \frac{1-F(q^*)}{\mu} \frac{l}{h} < 1-F(q^*).$$

Again, continue to assume $\mu > l/h$. The FRM welfare is as profit, that is $\mu h \mathbb{E}[q]$. Rewrite it as

$$\mu h \mathbb{E}[q \mid q \leq q^*] F(q^*) + \mu h \mathbb{E}[q \mid q \geq q^*](1-F(q^*))$$

The BOM welfare is

$$h\mathbb{E}[q \mid q \leq q^*] F(q^*) + \left(\frac{l}{h} h + \frac{h-l}{h} l\right) \mathbb{E}[q \mid q > q^*](1-F(q^*)).$$

Subtracting FRM from BOM we get

$$h(1-\mu)\mathbb{E}[q \mid q \leq q^*]F(q^*) - \left[\left(\mu - \frac{l}{h}\right)h - \frac{h-l}{h}l\right]\mathbb{E}[q \mid q > q^*](1-F(q^*)).$$

The first term is clearly positive. We now focus on when the second term of the difference. It is positive if

$$\left[\left(\mu - \frac{l}{h}\right)h - \frac{h-l}{h}l\right] > 0,$$

or

$$\mu > \frac{l}{h} + \frac{h-l}{h}\frac{l}{h}$$

It is now easy to see that when $\mu \leq \frac{l}{h} + \frac{h-l}{h}\frac{l}{h}$ then the welfare of BOM is higher than the welfare of FRM. To construct an example where the welfare of BOM is lower then FRM one assumes $\mathbb{E}[q \mid q \leq q^*]$ is sufficiently small and $\mu > \frac{l}{h} + \frac{h-l}{h}\frac{l}{h}$. $\square$