A lower-bound on monopoly profit for log-concave demand

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Abstract

If demand is log-concave a monopolist obtains at least $1/e$ of the area under the demand.
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There's a scattered literature, with contributions from both economists and computer scientists, that seeks to identify bounds to endogenous outcomes for relevant market models, assuming the analyst has only partial knowledge of the exogenous characteristics of those markets. For the case of a non-discriminating monopolist, and focusing on bounded demand functions, Neeman (2003) and Kremer and Snyder (2018) obtain a lower-bound on profit, while Condorelli and Szentes (2020) identify an upper bound to consumer surplus.[1]

In this note, it is shown that if the demand function is log-concave, then a zero marginal-cost monopolist who sets a uniform price will attain at least a fraction $1/e$ of the available gains from trade (i.e., the area under the demand curve). The class of log-concave distributions deserves attention because of its prominence in applied work. In the mechanism design literature, log-concave CDFs exhibit the useful increasing hazard rate property and guarantee that the monopolist's objective is well behaved. We refer to the classic Bagnoli and Bergstrom (2005) for more on log-concave functions.

We normalize the mass of consumers to one. Then, a demand function $D : \Re^+ \to [0, 1]$ is a non-increasing and left-continuous function mapping non-negative real numbers into the unit interval. Demand D is said to be *log-concave* if the function $\ln(D)$ is concave. Let

$$S(D) = \int_0^\infty D(x)dx$$

and

$$\Pi(D) = \sup_{p \in \Re^+} D(p)p$$

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[1] Condorelli and Szentes (2021) generalize the bounds above to Cournot competition, additionally characterizing all triples of producer surplus, consumer surplus and dead-weight loss that could arise. Other bounds in environments with Cournot monopoly and competition are obtained, for various different measures, in Anderson and Renault (2003), Johari and Tsitsiklis (2005), Tsitsiklis and Xu (2014) and Tsitsiklis and Xu (2013). As an example where the unknown is not the demand function, a full payoff characterization is offered by Bergemann et al. (2015) for a price-discriminating monopolist for a given demand function, assuming the analyst has no knowledge of the information available to said monopolist.

denote, respectively, the area under the demand curve (i.e., the first-best surplus available when all buyers buy) and the profit of the monopolist. Our result can now be stated.

Proposition 1 *If D is log-concave, then* $\dfrac{\Pi(D)}{S(D)} \geq \dfrac{1}{e}$.

The proof follows straightforwardly from two known inequalities, Lemma 5.4 in Lovász and Vempala (2007) and Lemma 1 in Condorelli and Szentes (2020).

Proof. Lemma 5.4 in Lovász and Vempala (2007) states that for any log-concave distribution of a real-valued random-variable X, then

$$\Pr\{X \geq \mathbb{E}[X]\} \geq \frac{1}{e}.$$

Noting that if D is log-concave, then $1 - \lim_{x^0 \to x^-} D(x^0)$ is a log-concave CDF and using the fact that $\mathbb{E}[X] = \int_0^\infty 1 - F(x)dx$ when r.v. X is non-negative valued and has CDF equal to F, we conclude that

$$D(S(X)) \geq \frac{1}{e}. \tag{1}$$

Lemma 1 in Condorelli and Szentes (2020) implies that for all $x \in \Re^+$

$$\frac{\Pi(D)}{x} \geq D(x).$$[2]

When evaluated at $x = S(D)$, the above inequality gives

$$\frac{\Pi(D)}{S(D)} \geq D(S(D)). \tag{2}$$

Combining (1) and (2) concludes the proof. ■

As discussed, an alternative lower-bound on monopoly profit is obtained in the papers mentioned in the first paragraph (for an explicit formula see for instance Condorelli and Szentes (2021)). Letting u be the maximum consumer valuation, it is shown that

$$\Pi(D) \geq \frac{S(D)}{-W_{-1}(\frac{-S(D)}{u \times e})} \equiv \pi_S(D)$$

where W_{-1} is the lower branch of the Lambert W function.[3] The bound presented in this note is not vacuous. First, since $\pi_S(D) \to 0$ if $u \to \infty$, then, without knowledge of the maximal valuation, $\pi_S(D)$ provides no information.[4] Second, even if one is ready to make assumptions on u, the bound obtained in this note will still be above $\pi_S(D)$ for demand functions such that $S(D)$ is sufficiently small. For instance, assuming $u = 1$, $\dfrac{S(D)}{e}$ is strictly below $\pi_S(D)$ for $0 < S(D) < e^{2-e} \sim 0.487$.

[2] Interestingly, this result represents an improvement on the Markov inequality, since for a non-negative r.v. X with CDF F it implies $\Pr\{X \geq x\} \leq \Pi(D^*(F))/x \leq \mathbb{E}[X]/x$, with $D^*(F)(x) = 1 - \lim_{x^0 \to x^-} F(x^0)$.

[3] While it cannot be expressed in terms of elementary functions it is defined by $W_{-1}(xe^x) = x$ for $x \leq -1$.

[4] In Condorelli and Szentes (2021) the bound is tight and is achieved by a truncated Pareto distribution, which of course is not log-concave.

To conclude, we present a straightforward application of the result. In McAfee and McMillan (1987) it is shown that if bidders in an auction, whose values are all independently drawn from the same log-concave distribution, must sustain an entry-cost greater than $1/e$, then a seller prefers to have just one bidder in the auction, rather than designing an auction that would allow two or more bidders to participate. Our result suggests an improvent on McAfee and McMillan (1987)'s bound. Since a monopolist can always attain $1/e$, it follows that as long as the entry cost is above $\frac{1-1/e}{2}(< 1/e)$ then the seller prefers a single buyer.

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